Replica trick to calculate means of absolute values: applications to stochastic equations

1991 J. Phys. A: Math. Gen. 24 4969

(http://iopscience.iop.org/0305-4470/24/21/011)

J. Phys. A: Math. Gen. 24 (1991) 4969–4979. Printed in the UK

Replica trick to calculate means of absolute values: applications to stochastic equations

J Kurchan

Dipartimento di Física, Università di Roma, La Sapienza, I-00185 Roma, Italy and INFN Sezione di Roma, Roma, Italy

Received 5 March 1991, in final form 14 June 1991

Abstract. A version of the replica trick can be used to evaluate means of the absolute values of functions that are not positive-definite (and with some modifications the mean logarithm of the absolute value). This trick can be applied in the context of stochastic equations, where the absolute value of the Jacobian of the stochastic function in each solution must be computed. The calculation of the average number of solutions of the 'naive' mean field equations for the SK spin-glass is discussed, an extension of the considerations to other mean field equations (such as TAP) is straightforward. The BRST supersymmetry is found to provide a useful tool in uncovering some puzzling aspects in these calculations. The trick presented here provides a possible solution to these problems.

1. Introduction

In this article we show that a version of the replica trick can be used to calculate the mean of the absolute value of quantities that are not positive-definite and, with slight modifications, the mean logarithm of the absolute value. Given a non-positive function $f(J)$ (J a finite-dimensional variable) we wish to calculate

$$\langle |f| \rangle_J = \int_{\mathcal{D}} \mathrm{d}J \, P(J) |f(J)| \tag{1}$$

where $P(J)$ is a probability distribution, and $\mathcal{D}$ is the (eventually infinite) domain in which $P \neq 0$.

This problem appears, for example, in the treatment of systems in which one is led to consider 'partition functions' that are not positive-definite, such as directed polymers in random media [1].

Another context in which the problem arises naturally is that of stochastic equations with more than one solution. Given a system of N equations on an N-dimensional variable $\boldsymbol{m}$:

$$G(\boldsymbol{m}, \boldsymbol{J}) = 0 \qquad i = 1, \ldots, N \tag{2}$$

where the $\boldsymbol{J}$ are some stochastic variables, and a function $F(\boldsymbol{m}, \boldsymbol{J})$ (which in this work we assume to be positive-valued) one can calculate the annealed mean of F over the solutions of (2):

$$\langle F \rangle_J = \left\langle \int \prod_{i=1}^{N} \mathrm{d}m_i \, \delta(G(\boldsymbol{m}, \boldsymbol{J})) F(\boldsymbol{m}, \boldsymbol{J}) |\det A| \right\rangle_J \tag{3}$$

where

$$A_{ij} = \frac{\partial G_i}{\partial m_j}. \tag{4}$$

If one disregards the absolute value, then F is weighted with the sign of the determinant of each solution [2]. If equations (2) have more than one solution, disregarding the absolute value is not a minor detail. For example, setting $F = 1$ integral (3) yields not the number of solutions of (2) but an invariant related to the contour values of G (for example, in one dimension either 1 or 0). Hence, the absolute value is mandatory to calculate the mean number of solutions.

In integral (3) *without the absolute value*, when exponentiating the Jacobian as an integral over Grassmann variables, the exponent has the fundamental BRST supersymmmetry discovered in this context by Parisi and Sourlas [2, 3] and Zinn-Justin [4]. This symmetry will be used later as a powerful tool with which to analyse the results obtained in such a calculation.

Both in the case of means of (or logarithms of) absolute values of stochastic functions and in the case of means over the roots of a stochastic equation, the trick consists in calculating the means with the function that is not positive-definite raised to even powers n. It is then proved that under very wide conditions there is a criterion to choose unambiguously an analytic continuation over n that yields the correct average. In the case of stochastic equations this can be viewed as a regularization that breaks the BRST symmetry.

In the second part of this article, we concentrate our attention on the calculation of the average number of solutions of the mean-field equations for the SK spin glass. For simplicity, the discussion is restricted to the 'naive' mean-field equations [5] (i.e. without the reaction term), but it can also be applied to the TAP equations [6]. In this part of the work we shall make extensive use of the results (and as far as possible the notations) of de Dominicis *et al* (DGGO) [7], Bray and Moore (BM) [8] and Takayama and Nemoto (TN) [9].

Using the BRST symmetry repeatedly we show that equation (3) (with $F = 1$) corresponding to the 'naive' mean-field equation for $T > 0$ yields one when computed without the absolute value (a similar argument can be made for the TAP equations).

We finally indicate how the trick discussed in the first section may provide a way out, and that the existing calculations may still be valid.

2. Means of absolute values

Consider first the integral (1). Suppose first that $\mathcal{D}$ is bounded, and define for $n = 2, 4, 6, \dots$

$$h(n) = \int_{\mathcal{D}} \mathrm{d}J\, P(J) f^n(J) \qquad n = 2, 4, \dots \tag{5}$$

(these are the magnitudes we actually calculate), and for all complex $n = n_\mathrm{R} + \mathrm{i} n_\mathrm{I}$, $n_\mathrm{R} > 0$

$$H(n) = \langle |f|^n \rangle_J = \int_{\mathcal{D}} \mathrm{d}J\, P(J) |f(J)|^n. \tag{6}$$

Clearly $h(n) = H(n)$ for n even. The question now is how to continue analytically $h(n)$ in order to reproduce $H(n)$. It turns out that this can be done unambiguously by imposing some growth conditions on the continuation: we show later that if we find a continuation $\bar{h}(n)$ of $h(n)$ that

(i) is analytic for $n_{\mathrm{R}} > 0$
(ii) satisfies for some positive reals C_1, Y_1:

$$|\tilde{h}(n)| < C_1 |Y_1^n| \qquad n_{\mathrm{R}} > 0 \tag{7}$$

(note that in general $\tilde{h}(1) \neq h(1)$), then;

$$\tilde{h}(n) = H(n) \qquad n_{\mathrm{R}} > 0 \tag{8}$$

and

$$\lim_{n \to 1} \tilde{h}(n) = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{J}\, P(\boldsymbol{J}) |f(\boldsymbol{J})| \tag{9}$$

(one can also calculate the mean logarithm of $|f|$). Any different continuation will violate the bound (7). In particular the one obtained by calculating (5) for all n grows exponentially in the imaginary direction if f has a region of positive expectation in which it is negative.

For bounded domain $\mathcal{D}$, and assuming f itself is bounded, $H(n)$ can be written as

$$H(n) = \int_0^{Y_2} \mathrm{d}y\, \hat{P}(y) y^n \tag{10}$$

where Y_2 is the maximum value of $|f|$ in $\mathcal{D}$, and $\hat{P}$ is a positive weight function ($\hat{P}\delta y = \int_{\delta\Omega} P(J)\,\mathrm{d}J$, $\delta\Omega$ the domain where $y < |f| < y + \delta y$). Hence, $H(n)$ is a moment function† and it follows that it is analytic for $n_{\mathrm{R}} > 0$ and satisfies

$$\tilde{H}(n) < C_2 |Y_2^n| \qquad n_{\mathrm{R}} > 0. \tag{11}$$

Now, putting $Y = \max(Y_1, Y_2)$, the function

$$\phi(n) = Y^{-n}(H(n) - \tilde{h}(n)) \tag{12}$$

is analytic and bounded for $n_{\mathrm{R}} > 0$, and vanishes for n even real. Hence, by Carlson's theorem [11] it is zero for $n_{\mathrm{R}} > 0$, and the uniqueness of $\tilde{h}$ is proved.

Consider now $\boldsymbol{J} = \{J_{ij}\}$ and $P(J_{ij})$ a Gaussian distribution

$$P(J_{ij}) = \left[\frac{N}{2\pi J}\right]^{1/2} \exp\left(-\frac{N}{2J} J_{ij}^2\right). \tag{13}$$

In this case we have to change the form of $H(n)$ slightly to make it satisfy a bound such as (11). We can define for example an n-dependent probability distribution (for $n_{\mathrm{R}} > 0$)

$$P_n(J_{ij}) = \left[\frac{N}{2\pi J}\right]^{1/2} \exp\left(-\frac{nN}{2J} J_{ij}^2\right) \tag{14}$$

and modify (5) and (6) by using (14) for each n. The factor in the exponential appropriately reduces the growth with n, provided that, for all J_{ij},

$$\frac{N}{2J} J_{ij}^2 - \ln|f| > \ln Y_1 \tag{15}$$

a quite unrestrictive condition. Definition (14) has to be modified if the logarithm of the absolute value is to be calculated.

† This way of showing analyticity and bounds is borrowed from Mehta [10], who applied it in a different context.

2.1. Stochastic equations

Consider now the system of N stochastic equations (2). We assume that they have a non-infinite number of solutions for $\boldsymbol{m}^s(\boldsymbol{J})$, $s = 1, \ldots, R(\boldsymbol{J})$ for every $\boldsymbol{J}$. We wish to calculate the mean over these solutions of the positive function F. Again, for $\boldsymbol{J}$ taking values over a finite domain $\mathcal{D}$ we define

$$h(n) = \left\langle \int \prod_{i=1}^{N} \mathrm{d}m_i\, \delta(G(\boldsymbol{m}, \boldsymbol{J}))F(\boldsymbol{m}, \boldsymbol{J}) \det nA \right\rangle_J$$
$$= \left\langle \sum_{s=1}^{R(J)} F(\boldsymbol{m}_s, \boldsymbol{J}) |\det A|^{n-1} \right\rangle_J \qquad n = 2, 4, \ldots \tag{16}$$

and

$$H(n) = \left\langle \sum_{s=1}^{R(J)} F(\boldsymbol{m}_s, \boldsymbol{J}) |\det A|^{n-1} \right\rangle_J \qquad n_{\mathrm{R}} > 0. \tag{17}$$

Then

$$\lim_{n\to 1} H(n) = \left\langle \sum_{s=1}^{R'(J)} F(\boldsymbol{m}_s, \boldsymbol{J}) \right\rangle \tag{18}$$

where the sum ranges over the $R'(J)$ solutions for which the Jacobian does not vanish.

Following very similar reasoning to that used previously, one shows that an analytic continuation $\tilde{h}(n)$ of $h(n)$ satisfying the bounds (7) coincides with $H(n)$ for $n_{\mathrm{R}} > 1$. For the case in which the $\boldsymbol{J}$ are Gaussian-distributed one can still prove the result by redefining the probability distribution as in (14), and the condition (15) now reads

$$\frac{N}{2J} J_{ij}^2 - \max_{\boldsymbol{m}_s} \ln |\det A(\boldsymbol{m}_s, \boldsymbol{J})| > \ln Y_1. \tag{19}$$

As usual, it is convenient to exponentiate the factors in (16)

$$h(n) = \left\langle \int_{x_i=-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_i \frac{\mathrm{d}x_i}{2\pi \mathrm{i}} \int \prod_i \mathrm{d}m_i \int \prod_{i,\alpha} \mathrm{d}\eta_{\alpha i}^* \,\mathrm{d}\eta_{\alpha i}\, F \right.$$
$$\left. \times \exp\left[\sum_i x_i G_i + \sum_{ij} \sum_{\alpha=1}^{n} \eta_{\alpha i}^* \eta_{\alpha j} A_{ij} \right] \right\rangle_J \tag{20}$$

where we have introduced the Grassmann variables $\eta_{\alpha i}^*$, $\eta_{\alpha i}$. As is well known [2–4] the exponent of (20) possesses, for $n = 1$, the supersymmetry

$$\delta m_i = \varepsilon \eta_i \qquad \delta \eta_i^* = -\varepsilon x_i. \tag{21}$$

For n even, the naive extension of this supersymmetry fails, while on the other hand there is the *continuous* 'replica' symmetry

$$\eta_{\alpha i} \to U_{\alpha\beta} \eta_{\beta i} \qquad \eta_{\alpha i}^* \to U_{\alpha\beta} \eta_{\beta i}^* \qquad UU^{\mathrm{T}} = 1. \tag{22}$$

Moreover, if the G_i are a gradient, then this symmetry is enlarged to

$$\varphi_i \to U\varphi_i \tag{23}$$

where φ_i is a $2n$-dimensional Grassmann vector with components $\eta^*_{\alpha i}$, $\eta_{\alpha i}$ and U a $2n \times 2n$ symplectic matrix:

$$U^{\mathrm{T}} Z U = Z \qquad U^{\dagger} U = 1 \tag{24}$$

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{25}$$

At this point one could wonder if it is possible to calculate $h(n)$ for even *negative* values of n, and then extend it to $n = 1$. This could, in principle, be achieved by expressing the negative powers of the determinant as an ordinary Gaussian integral (replicated 4, 8, ... times). The integration path should be chosen with care, otherwise the Gaussian integral diverges for Hessian matrices with negative (or zero) eigenvalues. If the Hessian matrix has negative and zero eigenvalues, then the function $H(n)$ will not, in general, be analytic on the whole real negative axis of n. Hence one does not find in that case a simple prescription to prove unicity. Moreover, exponentiating the determinant with the ordinary Gaussian integral one looses sight of the BRST symmetry for $n = 1$, which has profound consequences as we shall see in the next section.

3. The number of solutions of the 'naive' mean-field equations for the SK spin glass

In this section we discuss the calculation of the number of solutions of mean field equations for spin glasses, restricting ourselves for simplicity to the 'naive' ones, although the arguments are the same for the TAP equations. We prove that, if the sign of the determinant of the stochastic equation is disregarded, the result is *one*. We do so first using purely geometric arguments, and then directly on the large-N integral approach. For this last case the BRST symmetry is fundamental.

This raises the question as to if and why the known results of BM and TN are correct. We show how the trick discussed previously can be applied in this case. For apparently different reasons from those discussed in the previous section, in both these calculations the determinant was calculated to negative half integer powers using a Gaussian representation. For the reasons discussed previously we do not follow this approach, but calculate it instead to even positive powers.

Even though the results of the previous section tell us how to calculate the number properly *for finite* N, we find the usual problems of commuting the large-N limit with an analytical continuation. This introduces the uncertainty of finding the correct continuation of the saddle point. We indicate the general form of a replica symmetry breaking the ansatz for this case, and specify some conditions it must meet. However, the problem of which is the correct continuation remains open.

3.1. Computation without the absolute value: consequences of supersymmetry

The naive mean-field equations are

$$G_i = \tanh^{-1} m_i - \beta \sum_j J_{ij} m_j = g(m_i) - \beta \sum_j J_{ij} m_j = 0. \qquad (26)$$

Let us first discuss in some detail what happens if one tries to calculate the mean number of solutions disregarding the absolute value. To do this we set up an integral (16) with $n = 1$ (and $F = 1$) for this case.

$$h(1) = \left\langle \int_{x_i=-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_i \frac{\mathrm{d}x_i}{2\pi\mathrm{i}} \int_{m_i=-1}^{m_i=1} \prod_i \mathrm{d}m_i \, \exp\left[\sum_i x_i G_i\right] \det(a_i \delta_{ij} - \beta J_{ij}) \right\rangle_J$$
$$a_i = (1 - m_i^2)^{-1}. \qquad (27)$$

As pointed out in [12], this integral yields an invariant, for example with respect to β, and hence is equal to *one* (for finite β). This can be seen by invoking Morse theory (see [13] for a clear presentation). Because the magnetizations m_i never reach $m = \pm 1$ for N, β finite, for given J_{ij} and N and for all β^* smaller than β the zeros of (26) are confined to a distance greater than, say, $\mathrm{d}(\beta)$ of the faces of the hypercube $-1 < m_i < 1$. Consider a smooth surface completely contained within the hypercube and at a distance smaller than $\mathrm{d}(\beta)$ from these faces. On this surface the field G_i has positive norm, and one can calculate its degree [13] on it. This degree is an integer (the number of roots with a positive-determinant Hessian minus the number of roots with a negative-determinant Hessian) the mean over the J_{ij} of which is exactly $h(1)$. Now, starting from $\beta^* = 0$ and going up to $\beta^* = \beta$, the degree cannot change (because by construction no zero of (26) can approach the surface). Hence $h(1) = 1$.

Later, we shall prove this result again in detail using supersymmetry arguments [2–4]. At this point following BM and DGGO we exponentiate the delta functions and the determinant as in (20), we average over the J_{ij} and partially uncouple the sites by introducing order parameters q, V, λ and R (in the standard notation of BM and TN), to get

$$h(1) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mathrm{i}\lambda}{2\pi\mathrm{i}} \frac{N^{1/2}\,\mathrm{d}V}{(2\pi)^{1/2}} \frac{\mathrm{d}(2\mathrm{i}N^{1/2}R)}{(2\pi)^{1/2}} \mathrm{d}q \, \exp\left[N\left(-\lambda q - \frac{V^2}{2} + 2R^2\right)\right]$$
$$\times \left[\int_{x_i=-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_i \frac{\mathrm{d}x_i}{2\pi\mathrm{i}} \int_{m_i=-1}^{m_i=1} \prod_i \mathrm{d}m_i \int \mathrm{d}\eta_i^* \, \mathrm{d}\eta_i \right.$$
$$\times \exp\left(\sum_i (a_i - 2\beta J R)\eta_i^* \eta_i + \frac{1}{2}\beta^2 J^2 q x_i^2 + x_i g(m_i) \right.$$
$$+ \lambda m_i^2 + \beta J V m_i x_i + \frac{\beta^2 J^2}{N}\left[\left(\sum_i \eta_i^* x_i\right)\left(\sum_j \eta_j m_j\right)\right.$$
$$\left.\left.\left. + \left(\sum_i \eta_i^* m_i\right)\left(\sum_j \eta_j x_j\right)\right]\right)\right]. \qquad (28)$$

As pointed out by Plefka [14], the last term in the exponent cannot be neglected. Indeed, here we do not do so because this amounts to breaking the supersymmetry by hand.

We now complete the uncoupling by performing an *odd Grassmann* Hubbard–Stratonovich transformation. The result, in terms of the four even and four odd-Grassmann order parameters $y_k \equiv (\lambda, q, V, R)$, $C_k \equiv (\rho^*, \rho, \mu^*, \mu)$ is:

$$h(1) = \int \prod_{k=1}^{4} \mathrm{d}C_k \, \mathrm{d}y_k \exp[N(\Sigma_1 + \Sigma_2)] \Sigma_1 = -\lambda q - \frac{V^2}{2} + 2R^2 + \rho^*\rho + \mu^*\mu \quad (29)$$

$$\exp \Sigma_2 = \int \frac{\mathrm{d}x}{2\pi \mathrm{i}} \int_{m=-1}^{m=1} \mathrm{d}m \int \mathrm{d}\eta^* \, \mathrm{d}\eta \, \exp \Big((a - 2\beta J R)\eta^*\eta + \frac{1}{2}\beta^2 J^2 q x^2 + x g(m) + \lambda m^2 + \beta J V m x - \beta J \eta^* (\rho^* m + \mu^* x) - \beta J (\rho x + \mu m)\eta \Big). \quad (30)$$

One can check that the 'microscopic' BRST transformation (21) induces a 'macroscopic' supersymmetry with respect to the order parameters:

$$D\Sigma_1 = D\Sigma_2 = 0 \quad (31)$$

where

$$D = -\frac{2\lambda}{\beta J}\frac{\partial}{\partial \mu} - (V + 2R)\frac{\partial}{\partial \rho} + \frac{2\mu^*}{\beta J}\frac{\partial}{\partial q} + \rho^* \left(\frac{\partial}{\partial V} - \frac{1}{2}\frac{\partial}{\partial R} \right) \quad (32)$$

and

$$D^2 = 0. \quad (33)$$

Equation (31) can be obtained by considering an infinitesimal change in the order parameters generated by (32), and making a change of variables such as (21) in (30) to show that Σ_2 is invariant. For this one has to use the fact that the surface terms at $m = \pm 1$ vanish, reflecting the fact that the zeros of (26) cannot reach these limits for finite β, N.

Now, given a function of the order parameters $B(y_k, C_k)$, one shows [4] by integrating by parts that the expectation value of its 'total BRST derivative' is zero:

$$\int \prod_{k=1}^{4} \mathrm{d}C_k \, \mathrm{d}y_k \exp[N(\Sigma_1 + \Sigma_2)] DB = 0. \quad (34)$$

From this we extract some consequences. Consider a change in β:

$$\frac{\partial h(1)}{\partial \beta} = \int \prod_{k=1}^{4} \mathrm{d}C_k \, \mathrm{d}y_k \exp(N\Sigma_1) \tilde{D} \exp(N\Sigma_2) \quad (35)$$

where

$$\begin{aligned}\tilde{D} &= 2q\frac{\partial}{\partial q} + V\frac{\partial}{\partial V} + R\frac{\partial}{\partial R} + \rho^*\frac{\partial}{\partial \rho^*} + \rho\frac{\partial}{\partial \rho} + \mu^*\frac{\partial}{\partial \mu^*} + \mu\frac{\partial}{\partial \mu} \\ &= 2\frac{\partial}{\partial q}\cdot q + \frac{\partial}{\partial V}\cdot V + \frac{\partial}{\partial R}\cdot R - \frac{\partial}{\partial \rho^*}\cdot \rho^* - \frac{\partial}{\partial \rho}\cdot \rho - \frac{\partial}{\partial \mu^*}\cdot \mu^* - \frac{\partial}{\partial \mu}\cdot \mu. \end{aligned} \quad (36)$$

Integrating by parts and using the second of (36) to show that the surface term vanishes:

$$\frac{h(1)}{\partial\beta} = -\int\prod_{k=1}^{4} \mathrm{d}C_k\,\mathrm{d}y_k \exp(N\Sigma_1)\exp(N\Sigma_2)\tilde{D}\Sigma_1. \tag{37}$$

But a short computation yields

$$\tilde{D}\Sigma_1 = \beta^{-1} D[\beta J\mu q + (V-2R)\rho] \tag{38}$$

and hence the variation of $h(1)$ is a total BRST derivative, and $h(1)$ is indeed independent of the temperature. Another consequence is shown by putting $B=\mu$ and ρ to obtain, respectively,

$$\langle\lambda\rangle = \langle V+2R\rangle = 0 \tag{39}$$

where averages are taken with respect to the integrand in $h(1)$. These equations are satisfied at the saddle point level by the Sherrington–Kirkpatrick and the Sommers solutions [BM, TN], but not by the Bray–Moore solution, which is hence seen to break the supersymmetry.

It is interesting to note that in this context the spin-glass transition appears as a supersymmetry-breaking transition.

It would seem that if a supersymmetry-breaking saddle point is found then the solution can be different from one. The paradox is solved by noting that in that case the prefactor multiplying the saddle point value vanishes to all orders in $1/N$, as we now proceed to show (note that 1 will not be 'seen' by an $1/N$ expansion times an exponential in N).

To obtain the expansion we write the variables as

$$y_k = y_k^0 + N^{-1/2}y'_k \qquad C_k = N^{-1/2}C'_k \tag{40}$$

where y_k^0 are the saddle point values, and substitute in (29) and (30). The $1/N$ series can be expressed as

$$\exp[NS_0]\sum_{r=0}^{\infty} N^{-r/2}\int\prod_{k=1}^{4}\mathrm{d}C'_k\,\mathrm{d}y'_k A_r(C',y') \tag{41}$$

where NS_0 is the constant saddle-point value. The BRST generator can be written

$$D = N^{1/2}D_0 + D_1 = -N^{1/2}\frac{2\lambda^0}{\beta J}\frac{\partial}{\partial\mu} - N^{1/2}(V^0+2R^0)\frac{\partial}{\partial\rho} + D_1$$
$$0 = D_0D_1 + D_1D_0 = D_0^2 = D_1^2. \tag{42}$$

Note that $D_o \neq 0$ if and only if the supersymmetry is broken. The invariance (31) reads

$$D_0A_0 = 0 \qquad D_0A_{r+1} + D_1A_r = 0 \qquad r \geqslant 1. \tag{43}$$

If $D_0 \neq 0$ one can write, with B a constant times ρ or μ

$$1 = D_0 B. \tag{44}$$

Then

$$\begin{aligned}\int \prod_{k=1}^{4} \mathrm{d}C'_k \, \mathrm{d}y'_k \, A_r &= \int \prod_{k=1}^{4} \mathrm{d}C'_k \, \mathrm{d}y'_k [D_0 B]^{r+1} A_r \\ &= \int \prod_{k=1}^{4} \mathrm{d}C'_k \, \mathrm{d}y'_k [D_0 B]^{r} (A_{r-1} D_1 B) \\ &= \ldots = \int \prod_{k=1}^{4} \mathrm{d}C'_k \, \mathrm{d}y'_k (D_0 B) A_0 (D_1 B)^r \\ &= \int \prod_{k=1}^{4} \mathrm{d}C'_k \, \mathrm{d}y'_k B (D_0 A_0)(D_1 B)^r = 0 \end{aligned} \tag{45}$$

where we have repeatedly used integration by parts, (42) and (43). Hence we conclude that either supersymmetry is unbroken and the result is one, or it is broken and the prefactor cancels to all orders.

Clearly, the existing computations of the number of solutions as they stand need some independent justification for β finite.

3.2. *The trick*

Let us see to what extent the trick discussed in the previous section can provide a way out. We first note that condition (19) is satisfied by the solutions of the 'naive' mean field (as well as the TAP) equations. Hence, *for finite* N one can calculate $h(n)$ for even n with the probability distribution (14) and continue in a unique manner to obtain the correct result.

Unfortunately, it seems that one can only perform the calculation of $h(n)$ in the large-N limit, and only then extend the result to $n = 1$ 'sensibly'. One can hope in this way to obtain the large-N limit of the desired analytic continuation. The question becomes uncertain, the only excuse being that this uncertainty is common to most replica calculations.

In the calculation of $h(n)$, the order parameter R and the four order parameters ρ^*, ρ, μ^*, μ become replicated. The continuous symmetry (24) becomes more explicit by defining the $2n$-dimensional odd Grassmann vectors ψ and ξ with components $(\mu_\alpha, \rho^*_\alpha)$ and $(\rho_\alpha, \mu^*_\alpha)$, respectively. The role of the order parameter R is played by the most general Hermitian, self-dual matrix T (containing $2n^2 - n$ real parameters):

$$T = T^\dagger \qquad Z T Z^{-1} = T^{\mathrm{T}}. \tag{46}$$

Up to leading order the exponent for $h(n)$ is $N(\Sigma_1 + \Sigma_2)$, where

$$\Sigma_1 = -\lambda q - \frac{V^2}{2} + \mathrm{tr}(T^2) + \xi^{\mathrm{T}} Z \psi \tag{47}$$

$$\begin{aligned}\exp \Sigma_2 = \int \mathrm{d}x \int_{m=-1}^{m=1} \mathrm{d}m \int \mathrm{d}\eta^*_\alpha \, \mathrm{d}\eta_\alpha \, \exp\big(&\tfrac{1}{2}\beta^2 J^2 q x^2 + x g(m) + \lambda m^2 + \beta J V m x \\ &+ \tfrac{1}{2}\varphi^{\mathrm{T}} Z(a + 2i\beta J T)\varphi + \beta J m \psi^{\mathrm{T}} Z \varphi + \beta J \cancel{x} \xi^{\mathrm{T}} Z \varphi\big)\end{aligned} \tag{48}$$

which can be easily seen to be left invariant by the transformations U of (24)

$$T \to UTU^{\dagger} \qquad \xi \to U\xi \qquad \psi \to U\psi. \tag{49}$$

Since matrices of the form T can be diagonalized by these transformations, it suffices as far as saddle points are concerned to consider diagonal T. If the symmetry (49) is unbroken, then T must be proportional to the identity. In this case, for finite n the integrations over odd-Grassmann order parameters ψ, ξ can only affect the prefactor (as noted in DGGO), so they can be 'set to zero' in the saddle point calculation. This was assumed in TN for their corresponding parameters P, Q, and is automatic in this framework (in their case it was not because those parameters where not odd-Grassmann).

Assuming replica symmetry, one can readily check that a 'naive' continuation of the saddle point equations from even n to $n = 1$ does yield the BM solution. A necessary (although certainly not sufficient) condition for this solution to be credible would be to exhibit a 'reasonable' continuation of the prefactor that does not vanish as $n \to 1$.

Another possible scenario (perhaps more satisfactory) would be that replica symmetry is broken, and the effect of the breaking subsists down to $n = 1$ (then this solution would be automatically different from the one obtained directly at $n = 1$). The simplest way to implement the continuous replica symmetry breaking is to consider a saddle point with m pairs of eigenvalues of T equal to $\mathrm{i}R_1$ and $(n - m)$ equal to $\mathrm{i}R_2$. The anzatz in TN (called GE there) seems to correspond to this scheme with $m = 3/2$.

All these questions clearly require further clarification. In any case, if the present computations are indeed correct for β finite, it is because the integral *with the absolute value* has implicitly been taken. Note that in such a case we are computing the number of *all* stationary points of the free energy, not only the local minima.

4. Conclusions

We have shown that for a wide class of functions calculating the average of even powers n, there is a simple criterion to choose the analytic continuation which yields the average of the absolute value as $n \to 1$. In the saddle point approximation this result bears the uncertainty associated with commuting analytic continuation with the large-N limit.

In the case of stochastic equations, the resulting replica problem possesses a continuous (as opposed to permutation) symmetry. On the one hand this then implies that for replica symmetry breaking one only needs to consider an ansatz for which the order parameter matrix is diagonal, on the other hand Goldstone modes in replica space will appear even at integer n.

We have discussed the existing calculations in the number of solutions in mean field models of spin glasses, showing they need some extra *justification*. The trick presented here provides a possible way to find an answer, although uncertainties arising from continuation after saddle point evaluation need further elucidation.

We have found that the introduction of odd-Grassman *order parameters*, although not relevant as far as computations of saddle points are concerned, greatly clarifies the underlying symmetries of the problem.

Acknowledgments

I wish to thank S Franz for useful discussions, and Professor G Parisi for suggestions particularly on the role of supersymmetry. This work originated from discussions with Professor M Virasoro and I would like to acknowledge his help.

References

[1] Cook J and Derrida B 1990 *J. Stat. Phys.* **61** 961
[2] Parisi G and Sourlas N 1982 *Nucl. Phys.* B **206** 321
[3] Parisi G and Sourlas N 1979 *Phys. Rev. Lett.* **43** 744
[4] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[5] Bray A, Sompolinsky H and Yu C 1986 *J. Phys. C: Solid State Phys.* **19** 6389
[6] Mezard M, Parisi P and Virasoro M A 1987 *Spin Glasses and Beyond* (Singapore: World Scientific)
[7] deDominicis C, Gabay M, Garel T and Orland H 1980 *J. Physique* **41** 923
[8] Bray A J and Moore A M 1980 *J. Phys. C: Solid State Phys.* **13** L469
[9] Takayama H and Nemoto K 1990 *J. Phys. C: Solid State Phys.* **2** 1997
[10] Mehta M L, Random Matrices 1967 (New York: Academic)
[11] Titmarsh E C 1939 *The Theory of Functions* (Oxford: Oxford University Press)
[12] Vertechi D and Virasoro M A 1989 *J. Physique* **50** 2325
[13] Doubrovine B, Novikov S and Fomenko A 1982 *Geometrie Contemporaine* vol 2, ch 14 (Moscow: Mir)
[14] Plefka T 1982 *J. Phys. A: Math. Gen.* **15** 1971